

Homological connectivity of random hypergraphs

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Abstract

We consider simplicial complexes that are generated from the binomial random 3-uniform hypergraph by taking the downward-closure. We determine when this simplicial complex is homologically connected, meaning that its zero-th and first homology groups with coefficients in \mathbb{F}_2 vanish. Although this is not intrinsically a monotone property, we show that it nevertheless has a single sharp threshold, and indeed prove a hitting time result relating the connectedness to the disappearance of the last minimal obstruction.

1 Introduction

A classical result of Erdős and Rényi [5] states that the random graph $G(n, p)$ becomes connected with high probability when p is approximately $\frac{\log n}{n}$. Bollobás and Thomason [2] subsequently proved a hitting time result: With high probability the random graph *process*, in which edges are added one at a time in random order, becomes connected at exactly the moment when the last isolated vertex disappears. The aim of this paper is to prove a similar result for two-dimensional simplicial complexes.

While the random graph $G(n, p)$ is defined in a canonical way – fix the vertex set $[n] := \{1, 2, \dots, n\}$ and let each pair of vertices be connected by an edge with probability p independently – there are different natural models of random two-dimensional simplicial complexes. The following model was introduced by Linial and Meshulam [10]: Start with the full one-dimensional skeleton on the vertex set $[n]$ and let each triple of vertices form a 2-simplex with probability p independently. For such a complex X , they consider the first homology group $H_1(X; \mathbb{F}_2)$ with coefficients in \mathbb{F}_2 and prove that the vanishing of this homology group has a sharp threshold at $p = \frac{2 \log n}{n}$.

In this paper, we consider two-dimensional complexes that arise from the binomial random 3-uniform hypergraph, in which each triple forms an edge with probability p independently, by taking the downward-closure. Either model might be considered natural. Linial and Meshulam construct their complexes “bottom up”; thus, all 1-simplices have to be added to avoid restricting the

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possible set of 2-simplices. On the other hand, our complexes are constructed “top down” in the sense that we first choose the 2-simplices and then take only those 1-simplices needed to make the resulting structure a valid simplicial complex. However, we keep all of the 0-simplices (vertices) even if they are not contained in any 1- or 2-simplices since they were integral to the initial construction of the random hypergraph. (Deleting isolated vertices would leave a hypergraph which is *not* distributed as the binomial random hypergraph.) We discuss some further models in Section 6.

Unlike the complexes defined by Linial and Meshulam, a simplicial complex generated by a 3-uniform hypergraph does not have to be topologically connected. Therefore, we shall call a complex X *homologically connected* if *both* its first and its zero-th homology group with coefficients in \mathbb{F}_2 vanish. This notion of connectivity will turn out *not* to be monotone increasing—adding simplices to a homologically connected complex might yield a complex that is not homologically connected. Nevertheless, we will show that homological connectivity has a single sharp threshold.

1.1 Definitions and model

A family X of non-empty finite subsets of a set V is called a *simplicial complex* if it is downward-closed, i.e. if every non-empty set A that is contained in a set $B \in X$ also lies in X . The elements of X of size $k + 1$ are called the k -*simplices* of X . If a complex has at least one k -simplex but no $(k + 1)$ -simplices, then we call the complex k -*dimensional* or a k -*complex*. In a slight abuse of terminology, we also use k -complex to refer to a simplicial complex with dimension smaller than k , i.e. with no k -simplices.

Every 3-uniform hypergraph $H = (V, E)$ generates a 2-complex \mathcal{H} by taking the downward-closure of E . More particularly, we will generate the hypergraph H , and thus also the corresponding complex, randomly. Let $H^3(n, p)$ denote the random 3-uniform hypergraph on the vertex set $[n] := \{1, 2, \dots, n\}$ in which every triple of vertices forms an edge with probability p independently. This is the random *binomial* model, but we will also need to consider the random *uniform* model $H^3(n, m)$, the random 3-uniform hypergraph on vertex set $[n]$ which has its edge set E chosen uniformly at random from $\binom{[n]}{3}$. We denote the corresponding random 2-complexes by $\mathcal{H}_p(n)$ and $\mathcal{H}_m(n)$, respectively. When the number of vertices is clear from the context, we will often omit n in these notations. By \mathcal{H}_p^{LM} and \mathcal{H}_m^{LM} we denote the random 2-complex obtained from \mathcal{H}_p and \mathcal{H}_m respectively by adding all 1-simplices in $\binom{[n]}{2}$. Thus, \mathcal{H}_p^{LM} is the random 2-complex that was considered by Linial and Meshulam [10].

We will usually consider the complex instead of the hypergraph and refer to its 2-simplices as *faces*, to its 1-simplices as *edges*, and to its 0-simplices as *vertices*. The set of vertices and edges forms the *shadow graph*.

Definition 1. A simplicial complex X is called *homologically connected*, abbreviated to *hom-connected*, if its zero-th and first homology groups with coefficients in \mathbb{F}_2 , denoted by $H_0(X; \mathbb{F}_2)$ and $H_1(X; \mathbb{F}_2)$, vanish. The zero-th homology group vanishing is equivalent to being topologically connected, i.e. the shadow graph is connected. By the equality of simplicial homology and cohomology, the first homology group vanishes if and only if the first cohomology group does, i.e. if every 1-cocycle is a 1-coboundary, which can be stated in the following way.

(H1) For every 0-1 function f_e on the edges of X that has an even number of 1s on the boundary of each face of X there is a 0-1 function f_v on the vertices of X such that f_e is 1 for precisely those edges whose end vertices have different values for f_v .

We call a 0-1 function on the edges of X *bad* if it contradicts property (H1), i.e., if it is even on the boundary of every face but is not induced by a 0-1 function on the vertices. This is the case if and only if the shadow graph has a cycle whose edge-values sum to an odd number.

The *support* of a 0-1 function is the set of edges mapped to 1.

Topological connectivity in a complex generated from a 3-uniform hypergraph is equivalent to vertex-connectivity of the hypergraph. The requirement for the complex to be topologically connected did not appear in [10] since in \mathcal{H}_p^{LM} all edges are automatically present, so topological connectivity is trivial.

We further note that in contrast to the model of Linial and Meshulam, in our model the property of being hom-connected is *not* a monotone increasing property—this is because the shadow graph is not automatically complete, and by adding a new face we may create a new cycle which is part of a bad 0-1 function.

It is therefore not obvious that hom-connectivity should have a threshold in \mathcal{H}_p or indeed that it does not exhibit several thresholds where it oscillates between being connected and disconnected. However, our results in this paper prove that there is in fact a single threshold.

1.2 Main results

Linial and Meshulam [10] proved that hom-connectivity of the random complex \mathcal{H}_p^{LM} undergoes a phase transition at $p = \frac{2 \log n}{n}$; our first main result proves the analogous result in \mathcal{H}_p . We will consider the asymptotic properties of $\mathcal{H}_p(n)$ as n tends to infinity, hence any unspecified asymptotics in the paper are with respect to n . We say that an event holds *with high probability*, abbreviated to *whp*, if it holds with probability tending to 1 as n tends to infinity.

Theorem 1. *Let ω be any function of n which tends to infinity as n tends to infinity. Then with high probability*

- $\mathcal{H}_p(n)$ is not homologically connected if $p = \frac{\log n + \frac{1}{2} \log \log n - \omega}{n}$;
- $\mathcal{H}_p(n)$ is homologically connected if $p = \frac{\log n + \frac{1}{2} \log \log n + \omega}{n}$.

Compared with \mathcal{H}_p^{LM} , the probability threshold at which the phase transition occurs differs by approximately a factor of 2. The reason for this difference is that the minimal obstruction in \mathcal{H}_p^{LM} is an edge which does not lie in any face, which by definition does not exist in our model, so our minimal obstruction is different.

Indeed, we prove a hitting time result; the process becomes connected at the moment when the last minimal obstruction disappears. In this case, the minimal obstruction, denoted M , is defined as follows.

Definition 2. A copy of M in a 2-complex \mathcal{H} is a face with vertices a, b, c in which the edges ab and ac are in no other faces, and in which there is a path P_{ab} of edges between a and b which does not use the edges ab or ac .

In this case a bad function f would take the value 1 on ab and ac and 0 everywhere else. Since ab and ac are in no further faces, every face is even. However, P_{ab} together with the edge ab would form a cycle with precisely one value 1, ensuring that f cannot be generated by a vertex bipartition.

We write $M \subset \mathcal{H}$ if \mathcal{H} contains such a structure. We say that a certain face forms an M if it can be chosen as the face abc in an M . (Note that this is a slight abuse of terminology, since it also requires the existence and the non-existence of some other faces.)

For a hitting time result, we are concerned with the random hypergraph *process* in which at each time step we add a face chosen uniformly at random from among those faces not already present. (And we then consider the complex generated by the hypergraph at each time step.) At time m , this gives us the random uniform hypergraph $H^3(n, m)$. However, many calculations are easier in the random binomial hypergraph $H^3(n, p)$, where $p = m/\binom{n}{3}$.

Recall that $\mathcal{H}_m(n)$ denotes the complex generated by $H^3(n, m)$ and write $(\mathcal{H}_m(n))$ for the corresponding process.

Theorem 2. *With high probability, $(\mathcal{H}_m(n))$ is not homologically connected until the moment when the last copy of M disappears, and is homologically connected thereafter. The change happens at time*

$$m = \frac{n^2}{6} \left(\log n + \frac{1}{2} \log \log n + O(1) \right).$$

1.3 Paper overview and proof methods

This paper is laid out as follows.

In Section 3 we will determine when the last minimal obstruction disappears. Note that the presence of copies of M is not monotone, though we will show that at around the threshold determined in the introduction, whp the complex becomes M -free and no copy of M will appear for the rest of the process.

In Section 4 we will prove the subcritical cases of Theorems 1 and 2. The strategy is to divide the subcritical range into five subintervals

- $I_0 := [0, p_T)$
- $I_1 := [p_T, p_1)$
- $I_2 := [p_1, p_2)$
- $I_3 := [p_2, p_3)$
- $I_4 := [p_3, p_M)$.

Here p_T is the birth time of the face which causes the complex to become topologically connected and p_M is the birth time of the face which causes the last copy of M to disappear, which we prove to be

$$p_T = (2 + o(1)) \frac{\log n}{n^2}$$

and

$$p_M = \frac{\log n + \frac{1}{2} \log \log n + O(1)}{n}$$

whp (see Corollaries 11 and 8, respectively), while $0 < p_T < p_1 < p_2 < p_3 < p_M$ are chosen appropriately. Clearly in the interval $[0, p_T)$ the complex is not topologically connected and therefore not hom-connected. We then prove that whp there are four copies of M , called M_i for $1 \leq i \leq 4$, such that M_i exists in the complex throughout the interval I_i (Lemmas 12 and 13). Together, these intervals cover the entire range $[0, p_M)$ and resolve the subcritical case.

We note that while it may be possible to reduce the number of intervals by reducing the number of copies of M we use to cover the range from p_T to p_M , we certainly cannot expect just one copy of M to suffice. This is because to push the argument all the way to p_M we must certainly pick the very last copy of M to disappear. Having no choice about which copy of M we can choose means we cannot expect it to have existed for a very long time.

The supercritical case of Theorems 1 and 2 will be proved in Section 5. Since whp the complex is both topologically connected and contains no M in this range, it remains to prove that whp there are no larger obstacles to hom-connectivity, i.e. bad 0-1 functions with support of size at least 3. We consider a minimal bad support and observe important properties that the minimality guarantees (Lemma 14). Weaker versions of these properties were already considered by Linial and Meshulam. For example, a minimal bad support has to be connected. However, for our proof to work we need an additional property which we call *super-connectedness* (Definition 3), which also guarantees the existence of certain faces related to this support.

With this new definition, we prove the supercritical case in two ranges of the size k of the support. First for $3 \leq k \leq \log n$, we have a simple application of the first moment method (Lemma 15). We then prove the case when $k \geq \log n$, for which we need to bound the total number of possible bad supports more cleverly than in the case of small k , which we do via a breadth-first search process (Lemma 17). This search process, which is the second point in the supercritical case where our proof differs from that of [10], allows us to track the construction of a super-connected support and thus count the number of possibilities much more precisely than other methods allow us to.

2 Preliminaries

2.1 Intuition: Where “should” the threshold be?

We first justify our definition above of the minimal obstruction M . Of course, in one sense the smallest obstruction is an isolated vertex, but this is rather an obstruction to *topological* connectivity, and we expect the complex to become topologically connected well before it becomes hom-connected. We therefore assume topological connectivity and consider what the minimal obstruction to hom-connectivity should be.

We need a 0-1 function on edges with no odd faces, and since each edge lies in at least one face, this automatically means the support must have size at least two.³ The structure M described above gives rise to just such a function, as previously described.

³Note that this is the reason why the hom-connectivity threshold in this model is different to the threshold for the Linial & Meshulam model in [10], in which the minimal obstruction is an edge which does not lie in a face—in our model, by definition, such an edge does not exist.

Note that the property $M \subset \mathcal{H}$ is also *not* a monotone property—it demands the existence of a face and a path but also the non-existence of other faces. This will make various arguments slightly more tricky. However, intuitively the hypergraph process will initially not be hom-connected and copies of M will appear before it becomes hom-connected. As more faces are added, the copies of M will become the only obstructions to hom-connectivity and eventually when the last copy disappears, the hypergraph becomes hom-connected and remains so. We will prove that this intuition is correct in the course of the paper.

Let us provide a rough argument for why the threshold for the disappearance of the last copy of M should be at about $p = \frac{\log n + \frac{1}{2} \log \log n}{n}$.

First consider when the path P_{ab} is likely to appear. The probability that a fixed edge exists is approximately $1 - (1 - p)^n \simeq np$ (if this is small). There are $\Theta(n^{k-1})$ possible paths of length k , and so the expected number of these is $\Theta(n^{k-1}(np)^k)$. This is constant when $p = n^{-\frac{2k-1}{k}}$, so we can expect a constant length path to exist if $p \geq n^{-2+\delta}$, for some small constant $\delta > 0$. Note that this bound is significantly smaller than the $p = \Theta(\frac{\log n}{n})$ that we will predominantly be considering.

Next, consider when a face with two edges contained in no other face exists. The probability of three arbitrary vertices forming such a face is approximately $3p(1 - p)^{2n} \simeq 3pe^{-2pn}$, so the expected number of these is of order n^3pe^{-2pn} .

To determine the threshold (asymptotically approximately), we seek p such that $n^3pe^{-2pn} = 1$. This holds when

$$3 \log n + \log p - 2pn = 0$$

which implies

$$\begin{aligned} p &= \frac{3 \log n + \log p}{2n} \\ &= \frac{3 \log n + \log \left(\frac{3 \log n + \log p}{2} \right) - \log n}{2n} \\ &= \frac{\log n + \frac{1}{2} \log \log n + O(1)}{n} \end{aligned}$$

and so we expect a phase transition around $p = \frac{\log n + \frac{1}{2} \log \log n}{n}$.

2.2 Basic facts and notation

We will often use the following standard result.

Lemma 3 (Chernoff Bound, see e.g. [10]). *Given a binomially distributed random variable X with mean μ and a real number $a > 0$,*

$$\begin{aligned} \Pr(X \geq \mu + a) &\leq \exp \left(-\frac{a^2}{2(\mu + a/3)} \right); \\ \Pr(X \leq \mu - a) &\leq \exp \left(-\frac{a^2}{2\mu} \right). \end{aligned}$$

To aid in the transition between the two models $H^3(n, p)$ and $H^3(n, m)$ of random hypergraphs (and thus also between the corresponding models \mathcal{H}_p and

\mathcal{H}_m of random complexes), we utilise the standard trick of birth times: For each triple of vertices, choose a number from $[0, 1]$ uniformly at random and independently for each triple. This will be the *birth time* of the corresponding face. Then for any probability p , the hypergraph consisting of those faces with birth time at most p is distributed as $H^3(n, p)$, while the hypergraph process $(H^3(n, m))$ can be obtained by ordering the faces by increasing birth time (with probability 1 no two faces have the same birth time).

With this point of view, we sometimes think of $H^3(n, p)$ (and correspondingly \mathcal{H}_p) as also being a process in which p is gradually increased from 0 to 1. We sometimes talk of taking a “union bound over p ” in a certain range—this makes little sense if we think of p as being able to take any value within the interval, but if we condition on the set of birth times then in fact we only consider p taking the value of all birth times within the appropriate interval, which is a discrete set.

Finally in order to transfer various results between models, we observe the following, which is a simple application of the Chernoff bound.

Claim 4. *Given any interval $[q_1, q_2] \subset [0, 1]$ (where q_1, q_2 may depend on n) if $(q_2 - q_1)\binom{n}{3} \rightarrow \infty$, then with high probability the number of birth times within $[q_1, q_2]$ is $(1 \pm o(1))(q_2 - q_1)\binom{n}{3}$.*

Proof. Let X be the number of birth times within $[q_1, q_2]$, which is distributed $\text{Bi}(\binom{n}{3}, q_2 - q_1)$. Let $\mu := (q_2 - q_1)\binom{n}{3} \rightarrow \infty$. Observe that by Lemma 3,

$$\Pr(|X - \mu| \geq \mu^{2/3}) \leq 2 \exp\left(-\frac{\mu^{4/3}}{3\mu}\right) \leq 2 \exp(-\mu^{1/4}) = o(1). \quad \square$$

We will apply this claim a bounded number of times without explicitly mentioning it, often with $q_1 = 0$. Since we will only apply it a bounded number of times we may use a union bound over all error probabilities of size $o(1)$ to ensure that the stated events still hold whp.

We also note that conditioned on an edge not being present at time $p = q_1$, the probability that it is present at time q_2 is $\frac{q_2 - q_1}{1 - q_1}$. Thus we may obtain \mathcal{H}_{q_2} from \mathcal{H}_{q_1} by sprinkling an additional probability of $\frac{q_2 - q_1}{1 - q_1}$. Since we will only ever want to consider such a situation with $q_1 = o(1)$, we often simply take $q_2 - q_1$ as an approximation for $\frac{q_2 - q_1}{1 - q_1}$. This will be valid since a lower bound on the sprinkling probability will be sufficient.

We ignore floors and ceilings when this does not significantly affect the argument.

3 Minimal obstructions

In this section we prove various results related to when copies of M exist in \mathcal{H}_p . In particular, let p_M^* be the first birth time larger than $\frac{\log n + \frac{1}{4} \log \log n}{n}$ such that \mathcal{H}_p contains no copy of M , and recall that p_M is the time at which the last copy of M disappears.

Note that in theory we could have $p_M^* = \frac{\log n + \frac{1}{4} \log \log n}{n}$ if at this time there are no copies of M , though the results of this section show that whp this does

not happen. Our goal is to show that in fact whp

$$p_M^* = p_M = \frac{\log n + \frac{1}{2} \log \log n + O(1)}{n}.$$

For the rest of the paper, let us fix some constant $0 < \varepsilon < \frac{1}{10}$. (We think of ε as being arbitrarily small, but any constant in this range will be sufficient.) We will need the following basic fact—it tells us that by the time we have probability $p = n^{-1-\varepsilon}$, the shadow graph is highly connected.

Lemma 5. *Let $p = n^{-1-\varepsilon}$. Then with high probability, every pair of vertices is connected by at least \sqrt{n} paths of length 2 in the shadow graph of \mathcal{H}_p .*

Proof. Fix two vertices x and y and consider the number of paths of length 2 connecting them. To ensure independence in various calculations, we will only count paths of a certain type, which gives us a lower bound on the total number of paths. To this end, we pick disjoint vertex sets U and Z not containing x or y and of size $n/3$. We will count paths xzy where $z \in Z$ and the edges xz and zy exist because there are faces uxz and vzy with $u, v \in U$. Note that for fixed x and y , all faces which we consider are distinct, ensuring independence. Let X be the number of such paths xzy .

Now, the probability that a vertex z is the midpoint of such a path is equal to the probability that there are $u, v \in U$, not necessarily distinct, with uxz and vzy both being faces of the hypergraph. This probability is

$$(1 - (1 - p)^{n/3})^2 \geq (pn/3 - (pn/3)^2/2)^2 \geq n^{-3\varepsilon}.$$

This probability is independent for each z , so the number of paths we obtain dominates $\text{Bi}(n/3, n^{-3\varepsilon})$, and by Lemma 3 the probability that this is less than \sqrt{n} is at most

$$\exp\left(-\frac{(n^{1-3\varepsilon}/3 - \sqrt{n})^2}{2n^{1-3\varepsilon}/3}\right) \leq \exp\left(-\frac{n^{1-3\varepsilon}}{7}\right) \stackrel{(\varepsilon < \frac{1}{6})}{\leq} e^{-\sqrt{n}}.$$

Thus we may take a union bound over all $\binom{n}{2}$ possible choices for x and y and the probability that the statement in the lemma does not hold is at most

$$\binom{n}{2} e^{-\sqrt{n}} \leq e^{-n^{1/3}} = o(1)$$

as required. \square

Lemma 5 tells us that the paths necessary for an M are very likely to exist. This motivates the following definition, which is a relaxation of M : Let M' consist of a face with two edges contained in no other face (i.e. a copy of M , but without the requirement of having an additional path in the shadow graph). Clearly if $M' \not\subset \mathcal{H}$, then also $M \not\subset \mathcal{H}$. We will usually consider M' in a range of p where the existence of paths is extremely likely, so the existence of M' and M are essentially equivalent events (though we will only ever use the bound in the correct direction).

We next prove that in the range shortly before the critical threshold for hom-connectivity, the expected number of obstructions is concentrated around

its mean. (This result is stronger than we need for this section, but the stronger version will be necessary later on.) To help with this we talk of *rooted triples* forming a copy of M' . A rooted triple is a triple of vertices x, y, z in which one of these vertices (say x) is the root. We say that this rooted triple forms a copy of M' if the triple forms a face and xy and xz are in no other faces.

Lemma 6. *Let ω be any function of n which tends to infinity, let $\frac{1}{n \log n} \leq p \leq \frac{\log n + \frac{1}{2} \log \log n - \omega}{n}$ and let X be the number of rooted copies of M' in \mathcal{H}_p . Then with high probability, $X \sim \mathbb{E}(X) \geq \frac{n^3 p}{3} \exp(-2pn)$.*

Proof. We assume without loss of generality that $\omega = o(\log \log n)$. We start by approximating the first moment. We have

$$\mathbb{E}(X) = \binom{n}{3} 3p(1-p)^{2(n-2)} \geq (1+o(1)) \frac{n^3 p}{2} \exp(-(p+p^2)2n) \quad (1)$$

and in particular the desired lower bound follows since $p^2 n = o(1)$. Let us note here that the expectation is maximised at $p = \frac{1}{2n-3}$, and so minimised at either ends of the range of p . It is simple to check that the upper extreme of p gives the smaller expectation, which we bound by

$$\begin{aligned} \frac{n^3 p}{3} \exp(-2pn) &\geq (1+o(1)) \frac{n^2 \log n}{3} \exp(-2 \log n - \log \log n + 2\omega) \\ &= \frac{1+o(1)}{3} e^{2\omega} \rightarrow \infty. \end{aligned}$$

We also need to calculate the second moment. To do this we calculate the probability that two rooted triples of vertices both form minimal obstructions, distinguishing across the size of their intersection, showing that the probabilities are of similar order regardless of the intersection. Since almost all pairs of triples do not intersect, this will show that $\mathbb{E}(X^2)$ is dominated by the non-intersecting pairs of triples, as required.

The contribution to $\mathbb{E}(X^2)$ made by rooted triples which are the same except possibly the root is at most $9\mathbb{E}(X) = o(\mathbb{E}(X)^2)$. If the two triples are not the same, then we certainly require 2 faces. We claim that we also require $4n - O(1)$ non-faces in all cases. This is certainly clear in the cases when the intersection has size at most 1, since each of four edges must lie in $n - O(1)$ non-faces, and we can only double-count faces containing two of these, of which there are at most $\binom{4}{2}$. On the other hand, if the intersection has size 2, i.e. the two triples intersect in an edge, then this edge must certainly be in these two faces, and the remaining four edges are the ones which are in no further faces. Thereafter, we argue as before.

Now we note that $(1-p)^{O(1)} = 1 - o(1)$, and so the probability of two rooted triples (in which the triples are not identical) forming two copies of M' is approximately the same regardless of their intersection, and in particular approximately asymptotically the square of the probability of one rooted triple forming a copy of M' .

Thus the expected number of pairs of copies of M' is

$$\left(9 \binom{n}{3}^2 - O(n^5) \right) (1+o(1)) p^2 (1-p)^{4n-O(1)} = (1+o(1)) \mathbb{E}(X)^2.$$

Thus $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = o(\mathbb{E}(X)^2)$ and by Chebyshev's inequality we have $X \sim \mathbb{E}(X)$ whp, as required. \square

Lemma 7. *Whp for all $p \geq p_M^*$, \mathcal{H}_p contains no copy of M' , and therefore also no copy of M , i.e. $p_M = p_M^*$.*

Proof. We begin by observing that, conditioned on the high probability event of Lemma 5, a copy of M' can only appear if there are two incident pairs which are both not in \mathcal{H}_{p_M} , but such that the triple containing both of them is born as a face before any triple containing either one or the other.

We therefore first bound the number of pairs of incident pairs not in \mathcal{H}_p , for $p = \frac{\log n + \frac{1}{2} \log \log n}{n}$. The probability that two incident pairs are both not in \mathcal{H}_p is $(1-p)^{2n-3} \leq (1+o(1))e^{-2pn} = O\left(\frac{1}{n^2 \sqrt{\log n}}\right)$. Therefore the expected number of such pairs is $O\left(\frac{n}{\sqrt{\log n}}\right)$ and by Markov's inequality, whp there are at most $\frac{n}{\sqrt[3]{\log n}}$ of them.

Given such a pair, the probability that the face containing both of them is born before any face containing just one is of order $1/n$. Therefore the expected number of times a copy of M' is created throughout the rest of the process is $O(1/\sqrt[3]{\log n})$, and so whp none are created, as required. \square

Corollary 8. *With high probability, $p_M = \frac{\log n + \frac{1}{2} \log \log n + O(1)}{n}$.*

Proof. Whp we have $p_M \geq \frac{\log n + \frac{1}{2} \log \log n - \omega}{n}$ for any $\omega \rightarrow \infty$ by Lemma 5 and Lemma 6. On the other hand if $p = \frac{\log n + \frac{1}{2} \log \log n + \omega}{n}$, then the expected number of rooted copies of M' is

$$\begin{aligned} \binom{n}{3} 3p(1-p)^{2(n-2)} &\leq (1+o(1)) \frac{n^3 p}{2} \exp(-2pn) \\ &= (1+o(1)) \frac{n^2 \log n}{2} \exp(-2 \log n - \log \log n - 2\omega) \\ &\leq \exp(-2\omega) = o(1) \end{aligned}$$

and so by Markov's inequality, whp $p_M^* \leq \frac{\log n + \frac{1}{2} \log \log n + \omega}{n}$, and by Lemma 7 we have $p_M^* = p_M$ whp, completing the argument. \square

4 The subcritical case

In this section we will prove the first statements of Theorems 1 and 2. Unlike many other similar results on connectivity in random graphs or hypergraphs, the subcritical case is far from trivial. The reason for this is that hom-connectivity is not a monotone property, and therefore it is not enough to prove that \mathcal{H}_p is not hom-connected at time $p = \frac{\log n + \frac{1}{2} \log \log n - \omega}{n}$. Rather we have to prove that whp \mathcal{H}_p is not hom-connected at every p up to and including this one.

We begin by proving that whp \mathcal{H}_p becomes topologically connected at the moment when the last isolated vertex disappears, and that this occurs at around $p = \frac{2 \log n}{n^2}$. The proof is a simple adaptation of the corresponding result for graphs, and indeed follows as a special case of previously proved hypergraph results in [14] and [3], but we reprove it here for completeness.

Lemma 9. *Let $\delta > 0$ be constant. With high probability \mathcal{H}_p contains isolated vertices if $p \leq (2 - \delta) \frac{\log n}{n^2}$. In particular, \mathcal{H}_p is not topologically connected and therefore also not hom-connected.*

On the other hand, with high probability \mathcal{H}_p contains no isolated vertices if $p \geq (2 + \delta) \frac{\log n}{n^2}$.

Proof. We note that the presence of isolated vertices is a monotone decreasing property, therefore it suffices to prove each statement for the upper or lower bound on p respectively. For the first statement, let $p = (2 - \delta) \frac{\log n}{n^2}$. The probability that a vertex is isolated is

$$(1 - p)^{\binom{n-1}{2}} \geq e^{-pn^2/2 - p^2 n^2/2} \geq (1 + o(1))n^{-(1-\delta/2)}.$$

Thus if X is the number of isolated vertices, we have $\mathbb{E}(X) \geq (1 + o(1))n^{\delta/2}$. Furthermore, the probability that two distinct vertices are isolated is

$$(1 - p)^{2\binom{n-1}{2} - (n-2)} \leq e^{-p(n-2)^2} = n^{-(2-\delta)} e^{O(np)} \leq (1 + o(1))n^{-(2-\delta)}$$

and so

$$\mathbb{E}(X^2) \leq n(n-1)(1 + o(1))n^{-(2-\delta)} + \mathbb{E}(X) = (1 + o(1))\mathbb{E}(X)^2.$$

It follows that $\text{Var}(X) = o(\mathbb{E}(X)^2)$ and therefore by Chebyshev's inequality, whp there are isolated vertices as required.

The second statement simply follows from a first moment calculation: For $p = (2 + \delta) \frac{\log n}{n^2}$ we have

$$\mathbb{E}(X) = n(1 - p)^{\binom{n-1}{2}} \leq ne^{-pn^2/2 + pn^2/2} \leq (1 + o(1))n^{-\delta/2} = o(1)$$

so by Markov's inequality, whp there are no isolated vertices. \square

We call the components of the shadow graph *topological components*, and say that such a component is *trivial* if it consists of just one isolated vertex.

Lemma 10. *Let $\delta > 0$ be constant. With high probability, for all $\frac{\log n}{n^2} \leq p \leq (2 + \delta) \frac{\log n}{n^2}$, there is exactly one non-trivial topological component in \mathcal{H}_p .*

Let us note that the constant (i.e. 1) in the lower bound for p is not the optimal constant (which would in fact be $2/3$), but this result will be strong enough for our purposes and choosing this larger constant will make the proof significantly easier.

Proof. We first show that for $p = \frac{\log n}{n^2}$, whp there are no topological components of size k for $3 \leq k \leq n/4$ (note that a topological component of size 2 is not possible).

So consider the expected number of topological components of size k . There are at most $\binom{n}{k}$ ways of choosing the vertices, and there must be at least $\frac{k-1}{2} \geq \frac{k}{3}$ faces, with at most $\binom{k}{3}^{k/3}$ ways of choosing these $k/3$ faces. Finally, any triple containing at least one vertex from these k and one vertex from the remaining

$n - k$ cannot be a face. Thus there must be at least $k(n - k)(n - 2)/2$ non-faces. Thus the expected number of topological components of size k is at most

$$\begin{aligned}
\binom{n}{k} (1 - p)^{k(n-k)(n-2)/2} p^{k/3} \binom{k}{3}^{k/3} &\leq \left(\left(\frac{en}{k} \right)^3 (1 - p)^{3(n-k)(n-2)/2} p \frac{k^3}{6} \right)^{k/3} \\
&\leq \left(\Theta \left(n(\log n) e^{-3p(n-k)(n-2)/2} \right) \right)^{k/3} \\
&\stackrel{(k \leq \frac{n}{4})}{\leq} \left(\Theta \left(n(\log n) e^{-\frac{3}{2} \log n \cdot \frac{3}{4}(1+o(1))} \right) \right)^{k/3} \\
&\leq \left(\Theta \left(n(\log n) n^{-10/9} \right) \right)^{k/3} \\
&\leq n^{-k/30}.
\end{aligned}$$

Thus the probability that there is any topological component of size between 3 and $n/4$ is at most

$$\sum_{k=3}^{n/4} n^{-k/30} \leq n^{-1/11} = o(1).$$

Thus at time $p = \frac{\log n}{n^2}$, whp we have at most four non-trivial topological components, each of size at least $n/4$. We now show that within the time interval $\frac{\log n}{n^2} \leq p \leq (1 + \delta) \frac{\log n}{n^2}$, these components will merge together whp. The probability that two such components do not merge once we add the additional probability of (at least) $p' = \delta \cdot \frac{\log n}{n^2}$ is at most

$$(1 - p')^{\frac{n^2}{16} \cdot \frac{n-2}{2}} \leq \exp(-p' n^3 / 33) = \exp\left(-\frac{\delta}{33} n \log n\right) = o(1)$$

and since we have a bounded number of topological components, whp they all merge together.

Finally, we need to show that from time $p = \frac{\log n}{n^2}$ onwards we don't create any more non-trivial topological components. We observe that if any such component is created in the process, then at the time that it first becomes non-trivial it will have size 3. Therefore it is enough to show that whp, from time $p = \frac{\log n}{n^2}$ onwards no components of size 3 are created from previously isolated vertices.

Let us first consider the number of isolated vertices at time $p = \frac{\log n}{n^2}$. The expected number is

$$n(1 - p)^{\binom{n-1}{2}} \leq n \exp\left(-p \frac{(n-2)^2}{2}\right) = n \exp\left(-\left(\frac{1}{2} + o(1)\right) \log n\right) \leq n^{\frac{1}{2} + o(1)}.$$

Thus by Markov's inequality, whp there are at most $n^{3/5}$ isolated vertices.

Now conditioned on having at most $n^{3/5}$ isolated vertices at time $p = \frac{\log n}{n^2}$, let us denote the set of isolated vertices by X . To create a new topological component of size 3, a face containing three vertices of X would have to be born before any of the faces containing one of these vertices and vertices not from X . For any one vertex of $x \in X$, the number of faces containing x and two other vertices of X is at most $n^{6/5}$, and so the probability of one of these

faces being the next to be born is $O(n^{6/5}/n^2) = O(n^{-4/5})$. Taking a union bound over all $O(n^{3/5})$ vertices of X , the probability of creating any topological component of size 3 is at most $O(n^{-1/5}) = o(1)$. \square

From Lemmas 9 and 10 we immediately deduce the value of the birth time p_T of the face that makes the complex topologically connected.

Corollary 11. *With high probability, $p_T = (2 + o(1))\frac{\log n}{n}$.*

Recall that we split the range from p_T to p_M , the moment when the last copy of M disappears, into four intervals I_1, \dots, I_4 , and aim to show that whp for each $1 \leq i \leq 4$, there is a copy M_i of M which remains in place throughout the interval I_i . Also recall that ε is a fixed constant with $0 < \varepsilon < \frac{1}{10}$.

Lemma 12. *With high probability, at the moment the shadow graph becomes connected, the face which we just added forms a copy M_1 of M , and remains an M until $p_1 = n^{-1-\varepsilon}$.*

Proof. By Lemmas 9 and 10, whp the shadow graph becomes connected when the last isolated vertex disappears, and this occurs at time about $\frac{2 \log n}{n^2}$.

We claim that whp, at the moment the shadow graph becomes connected, we only connected one isolated vertex rather than 2. For conditioned on the moment when the number of isolated vertices becomes at most 2, if there is only one left, then the claim follows, but if there are two, then the probability that the next face containing one of them contains both is $\frac{n-2}{2\binom{n-2}{2}} = O(1/n)$.

Therefore whp we had just one isolated vertex x before adding the face $e = xyz$. Then yz is distributed uniformly at random in $V - x$, and the probability that the edge yz was already in a face is at most $np = O\left(\frac{\log n}{n}\right)$ (conditioned on the high probability event that $p \leq \frac{(2+\delta)\log n}{n^2}$).

However, y, z were in a connected component before we added the face e , therefore there is a yz -path in the shadow graph not using this face. Thus xyz forms a copy M_1 of M .

It remains to show that M_1 remains an M for a long time. This is certainly the case for as long as xy and yz are contained in no further faces. Sprinkling a further probability of (at most) $n^{-1-\varepsilon}$ means that the probability of finding another face containing one of these edges is at most $2n \cdot n^{-1-\varepsilon} = o(1)$, as required. \square

Lemma 13. *With high probability there are copies M_2, M_3, M_4 of M which remain in place for the following time ranges*

- M_2 for $n^{-1-\varepsilon} \leq p \leq \frac{\log n}{10n}$;
- M_3 for $\frac{\log n}{10n} \leq p \leq \frac{\log n}{n}$;
- M_4 for $\frac{\log n}{n} \leq p \leq p_M$ (i.e. M_4 is the last copy of M to disappear).

Proof. By Lemma 6, at time $p_2 = \frac{\log n}{10n}$ there are at least

$$\begin{aligned} \frac{n^3 p}{3} \exp(-2pn) &= \Theta\left(n^2 \log n \exp\left(-\frac{\log n}{5}\right)\right) \\ &\geq n^{9/5} \end{aligned}$$

copies of M' . By Lemma 5, whp for $p \geq n^{-1-\varepsilon}$, such a triple will always have been a copy of M provided the corresponding face exists. We therefore need to show that whp, at least one of these faces already existed at time $p_1 = n^{-1-\varepsilon}$.

To do this, observe that given that these faces exist at time p_2 , their birth times are uniformly distributed in $[0, \frac{\log n}{10n}]$. The probability that any fixed such face existed at time $p_1 \geq \frac{1}{n^{2\varepsilon}} \frac{\log n}{10n}$ is at least $1 - n^{-2\varepsilon}$. Thus, the probability that none of them existed at time p_1 is at most

$$(1 - n^{-2\varepsilon})^{n^{9/5}} \stackrel{(\varepsilon < \frac{1}{10})}{\leq} e^{-n^{8/5}} = o(1)$$

as required.

An essentially identical argument also shows that whp there is a minimal obstruction throughout $p \in [\frac{\log n}{10n}, \frac{\log n}{n}]$ (since at time $p_3 = \frac{\log n}{n}$ whp we have a growing number of copies of M by Lemma 6), and that the final minimal obstruction to disappear, at time $p_M = \frac{\log n + \frac{1}{2} \log \log n + O(1)}{n}$ already existed at time $p_3 = (1 - o(1))p_M$. \square

Together, Lemmas 12 and 13 prove the subcritical case of Theorem 2, and together with Corollary 8, this proves the subcritical case of Theorem 1.

5 The supercritical case

In this section we prove the supercritical cases of the two main theorems. We need the following definition.

Definition 3. An edge set in a 2-complex \mathcal{H} is called *super-connected* if it cannot be partitioned into two non-empty sets such that every face of \mathcal{H} has edges in at most one of the two sets.

Note that an alternative and equivalent definition of super-connectedness comes from considering an auxiliary graph G whose vertices are the edges of \mathcal{H} and with two such vertices connected by an edge if the corresponding edges lie in a common face of \mathcal{H} . Then a super-connected set of edges in \mathcal{H} corresponds to a set of vertices in G which induces a connected subgraph.

Lemma 14. *Let \mathcal{H} be an arbitrary 2-complex and let F be an edge set in \mathcal{H} that is the support of a bad 0-1 function and smallest possible with that property. Then*

- (i) *every vertex of \mathcal{H} has degree less than $\frac{n}{2}$ in F and*
- (ii) *F is super-connected.*

Let us note that a similar observation, but with only connected in place of super-connected, was in [10].

Proof. Suppose, for a contradiction, that F does not satisfy (i) and let v be a vertex of degree $d_F(v) \geq \frac{n}{2}$. Let E_v be the set of edges of \mathcal{H} at v and let F' be the symmetric difference of F and E_v , i.e. an edge of \mathcal{H} is in F' if and only if it either is in F and not incident with v or is incident with v and not in F . By construction, since F is the support of a bad 0-1 function, so is F' . But

$$|F'| = |F| - d_F(v) + (n - 1 - d_F(v)) < |F|,$$

contradicting the minimality of F . This proves (i).

Now suppose that F is not super-connected and let (F_1, F_2) be a partition witnessing this fact. By the minimality of F , both of the functions f_1, f_2 with support F_1, F_2 are not bad, i.e. each F_i either is odd on the boundary of some face or every cycle in \mathcal{H} meets F_i in even number of edges. Since f is bad, there is a cycle C in \mathcal{H} that meets F in an odd number of edges. Without loss of generality, C also meets F_1 in an odd number of edges. Since f_1 is not bad, there is a face σ whose boundary meets F_1 in an odd number of edges. By the choice of (F_1, F_2) , the boundary of σ avoids F_2 and thus meets F in an odd number of edges, a contradiction to the fact that f is bad. This proves (ii). \square

Lemma 15. *For $p = (1 + o(1)) \frac{\log n}{n}$, whp there is no bad 0-1 function on the edges of \mathcal{H}_p with super-connected support of size $3 \leq k \leq \log n$.*

Proof. For given $k \geq 3$ we calculate the number of ways of choosing a support of size k . Since the support must be connected, it covers at most $k + 1$ vertices and the number of ways of choosing it is at most

$$\binom{n}{k+1} \binom{\binom{k+1}{2}}{k} \leq \left(\frac{en}{k+1} \right)^{k+1} \left(\frac{e(k+1)}{2} \right)^k \leq (10n)^{k+1}$$

(This calculation was in [10].) There are at most k^{k-2} ways in which the support can be super-connected by $k-1$ faces with at least two edges in the support and each such face is present with probability p . In total, the probability that the chosen support is actually super-connected is at most $k^{k-2} p^{k-1}$. Each of the (at least) $k(n-k-1)$ triples with two vertices forming an edge in the support and the third being elsewhere is not allowed to be in \mathcal{H}_p . Therefore, the probability that a bad 0-1 function as in the claim exists is

$$\begin{aligned} P_1 &\leq \sum_{k=3}^{\log n} (10n)^{k+1} k^{k-2} p^{k-1} (1-p)^{k(n-k-1)} \\ &\leq \sum_{k=3}^{\log n} \frac{10n}{k^2 p} \left(10n k p e^{-(1+o(1)) \frac{\log n}{n} (n-k-1)} \right)^k \\ &\leq \sum_{k=3}^{\log n} n^2 \left(11k \frac{\log n}{n^{1+o(1)}} \right)^k \\ &\leq \sum_{k=3}^{\log n} n^2 \left(\frac{(\log n)^2}{n^{3/4}} \right)^k \\ &\leq (\log n) n^{-1/5} = o(1). \end{aligned} \quad \square$$

We note that a similar calculation for P_1 also works for k up to $n^{1-\varepsilon}$. However, we do not need this since we will cover this range with a different argument which we use for all large k .

For the case $k = 2$, the above calculation is not strong enough, but in this case we can simply apply Corollary 8. Furthermore, bad functions with support of size $k = 1$ are not possible, because every 0-1 function with support of size one has an odd face.

For very large k (larger than $\log n$), the bound above on the number of super-connected supports becomes very weak, so we will need a different way

of counting them (we use a breadth-first search). In particular, once k becomes larger than linear, the bound $k(n - k - 1)$ on the number of non-faces becomes useless and we also need a better way of counting these. For this, we quote the following result of Linial and Meshulam.

Proposition 16 ([10, Proposition 2.1]). *Let G be any graph on n vertices whose edges are the support of some smallest bad 0-1 function for a 3-uniform hypergraph. (So in particular, G has exactly one non-trivial component, maximum degree at most $n/2$ etc.) Let $B(G)$ be the number of bad triples, i.e. triples containing an odd number of edges of G .*

Then $B(G) \geq \frac{1}{120}|E(G)|n$.

For convenience we define $c := \frac{1}{120}$. With the help of this proposition we can prove the range when k is large. In this case the error probabilities are small enough that we may rule out a bad function not just for *some* p , but for *all* p large enough.

Lemma 17. *Whp for all $p \geq (1 + o(1))\frac{\log n}{n}$ there is no smallest bad 0-1 function on the edges of \mathcal{H}_p with support of size $k \geq \log n$.*

Proof. Recall that the support of a smallest bad 0-1 function must be super-connected (Lemma 14(ii)), and therefore we can discover it from an edge via a breadth-first search. More precisely, start from any edge of the support and query all triples containing it. Any triple which forms a face must have exactly one further edge of the support contained in it (otherwise it would be odd). From all further support edges found in this way, according to some arbitrary but pre-determined order, we continue the process (though querying only triples containing edges not yet known to be in the support). By the super-connectivity, we must find the whole support in this way.

Let us bound the number of components of size k which can be found by this process. From each edge we may query up to n triples, and suppose that from the i -th edge we find b_i faces in the BFS. The number of possible ways this can occur is at most $\binom{n}{b_i} 2^{b_i}$ (we choose which b_i faces are present, and for each face exactly one of the two further edges within it must be in the support). Thus conditioned on the sequence (b_i) , the total possible number of supports of size k is at most

$$\binom{n}{2} \prod_{i=1}^k \binom{n}{b_i} 2^{b_i} \leq \binom{n}{2} \frac{(2n)^{k-1}}{\prod_{i=1}^k b_i!}$$

where the inequality holds because the b_i must sum to $k - 1$. Furthermore, by Proposition 16 the probability that one fixed support exists and has no odd face is at most

$$p^{k-1}(1-p)^{ckn} \leq (p(1-p)^{cn})^{k-1}.$$

By differentiating the expression $p(1-p)^{cn}$, we can see that this expression has its maximum at $p = \frac{1}{1+cn} \ll \frac{\log n}{n}$. Therefore in the following calculations, we may substitute in the lower bound of $(1 + o(1))\frac{\log n}{n}$ for p . Thus the probability

p^* that some such support exists and has no odd face satisfies

$$\begin{aligned}
p^* \prod_{i=1}^k b_i! &\leq \binom{n}{2} (2np(1-p)^{cn})^{k-1} \\
&\leq \binom{n}{2} \left(3(\log n) e^{-(c+o(1)) \log n}\right)^{k-1} \\
&\leq \left(3(\log n) n^{-2c/3+2/(k-1)}\right)^{k-1} \\
&\leq n^{-ck/2}.
\end{aligned}$$

Here for the last inequality we used the fact that $k \geq \log n$.

However, we still need to sum over all possible sequences b_i . We now make a case distinction based on the number of b_i which are large. Let

$$\ell := |\{i : b_i \geq n^{c/4}\}|.$$

For fixed ℓ , we very crudely bound the number of possible sequences b_i by

$$k^\ell n^\ell (n^{c/4})^{k-\ell}.$$

(We choose which ℓ positions have $b_i \geq n^{c/4}$, for each of these we choose a b_i at most n , for all others we choose a b_i at most $n^{c/4}$.)

On the other hand, we have

$$\prod_{i=1}^k b_i! \geq ((n^{c/4})!)^\ell \geq (n^{c/4})^{\ell n^{c/5}} \geq n^{\ell n^{c/6}}.$$

Thus

$$\begin{aligned}
\sum_{b_1, \dots, b_k} \frac{1}{\prod_{i=1}^k b_i!} &\leq \sum_{\ell=0}^k \frac{k^\ell n^\ell (n^{c/4})^{k-\ell}}{n^{\ell n^{c/6}}} \\
&= n^{ck/4} \sum_{\ell=0}^k \left(\frac{kn}{n^{c/4} \cdot n^{n^{c/6}}} \right)^\ell \\
&\leq n^{ck/4} \cdot (k+1).
\end{aligned}$$

Combining this with our previous bounds, the probability that there exists a bad support of size k is at most

$$n^{-ck/2} n^{ck/4} \cdot (k+1) \leq n^{-ck/5}.$$

Summing over all $k \geq \log n$, the probability that any such bad support exists is at most $n^{-\frac{c}{5} \log n}$. This bound is valid for any single $p \geq (1+o(1)) \frac{\log n}{n}$, and taking a union bound over all $O(n^3)$ birth times in this range, we conclude that the probability that any bad minimal support of size at least $\log n$ ever appears in this range is at most

$$n^{-\frac{c}{7} \log n} = o(1)$$

as required. \square

We note here that there is nothing particularly special about the bound $k \geq \log n$. In fact, an identical argument works for k larger than some absolute constant which is related to c .

Together with the fact that there are no bad functions with support of size one, Corollary 8 and Lemmas 15 and 17 show that at time $p = \frac{\log n + \frac{1}{2} \log \log n + \omega}{n}$, whp \mathcal{H}_p is hom-connected. However, we still need to prove that whp it does not become disconnected again. We aim to do this by showing that a new obstruction cannot appear suddenly at a large size – rather, we must first see a copy of M , which by Corollary 8 we already know does not happen whp.

For supports of size at least $\log n$, this is already implied by Lemma 17. We proved this since the error probability was small enough that we could take a union bound over all remaining birth times,

Indeed, a similar argument would work for supports of size at least 6, but this calculation does not work for $k \leq 5$. Since for small supports we have to be more careful in any case, we use this argument for all sizes up to $\log n$.

For a 2-complex \mathcal{H} and a triple of vertices T , let $\mathcal{H} + T$ denote the complex obtained from \mathcal{H} by adding T as a face and (if necessary) all pairs in T as edges.

Lemma 18. *Suppose that in a 2-complex \mathcal{H} each pair of vertices is connected by at least k paths of length 2 in the shadow graph. Suppose further that \mathcal{H} is hom-connected, but for some triple T , $\mathcal{H} + T$ contains a bad 0-1 function with support of size at most k . Then $M \subset \mathcal{H} + T$.*

Proof. Let f be a bad 0-1 function in $\mathcal{H} + T$ with minimal support.

Suppose first that the support of f contains an edge e outside of T . Let S be the support of f and let S' be a maximal subset of S which contains e and is super-connected in \mathcal{H} . Let f' be a new 0-1 function whose support is exactly S' .

Now by the maximality of S , any face of \mathcal{H} meeting S' cannot meet $S \setminus S'$, and since such a face was even with respect to f , it is also even with respect to f' . On the other hand, since $|S' \setminus \{e\}| \leq k - 1 < k$, the edge $e \in S'$ is contained within a triangle of \mathcal{H} which contains no further edges of S' , and thus forms an odd cycle with respect to f' . But this means that \mathcal{H} is not hom-connected, which is a contradiction.

Thus the support of f is contained within T . But then (by the fact that T must be an even face) the support consists of exactly 2 edges and T forms a copy of M in $\mathcal{H} + T$. \square

We can now complete the proof of the supercritical case. For we know by Lemma 17 that whp the smallest obstruction is of size smaller than $\log n$ for any $p \geq \frac{\log n}{n}$. Furthermore, we know that at time $p = p_M > \frac{\log n}{n}$, \mathcal{H}_p is hom-connected whp by Lemma 15. Finally, by Lemma 5 \mathcal{H}_p satisfies the shadow graph condition of Lemma 18.

Let us condition on these high probability events all occurring. Now suppose for some $p \geq p_M$, \mathcal{H}_p is not hom-connected, and let p be minimal such that this is the case. Then p is the birth-time of a face T , and \mathcal{H} , the complex just before this face was born, satisfies all the conditions of Lemma 18. But then $\mathcal{H}_p = \mathcal{H} + T$ contains a copy of M , contradicting the fact that $p \geq p_M$.

6 Concluding remarks

6.1 Search processes for hypergraphs

In Section 5 we used a search process to allow us to better count the possible number of super-connected supports. Such search processes in hypergraphs have been used previously, for example in [4] and [11] to determine the threshold for high-order phase transitions in hypergraphs, inspired by previous work for graphs in [9].

6.2 Alternative models

There are several possible ways of generating a random 2-complex. If we start from a random binomial 3-uniform hypergraph, we must add some edges to ensure that we have a complex. The model of Linial and Meshulam and the model we consider in this paper lie at the two extremes—either adding in all possible edges, or only adding those edges we really have to. One might also consider what happens in between, if only *some* of the (not strictly necessary) edges are added, possibly randomly. However, as far as the hom-connectivity of the resulting complex is concerned, this is essentially covered by the results of [10] and this paper. Indeed, if any edge is not contained in a face, the complex is not hom-connected; otherwise we have the model we considered here.

It is also possible to construct a 2-complex from a graph rather than from a 3-uniform hypergraph by taking all triangles of the graph as faces. This is a special case of the *clique complex* which has been studied for example in [12] and [6].

6.3 Higher dimension

A natural question would be to ask whether the results in this paper extend to higher-dimensional complexes. For a k -complex generated from a $(k+1)$ -uniform hypergraph, we could ask whether the 0-th, 1st, \dots , $(k-1)$ -th homology groups all vanish.

For the analogue of the \mathcal{H}_p^{LM} model of Linial and Meshulam the obvious conjecture is that the $(k-1)$ -th homology group should vanish once every set of k vertices is contained in a k -simplex, which property has a threshold function of $p = \frac{k \log n}{n}$. This was indeed confirmed to be the threshold in [13]. The behaviour within the critical window was subsequently examined in [7].

In the analogue of our \mathcal{H}_p model, we would have to consider the vanishing of the j -th homology group separately for $j = 0, \dots, k-1$. It is not too hard to see that the vanishing of the zero-th homology group in the k -dimensional model has threshold $p = \frac{\log n}{\binom{n}{k}}$. For general j , we expect the threshold for the vanishing of the j -th homology group to be of order $\frac{\log n}{n^{k-j}}$.

6.4 Alternative definitions of connectivity

If X is a simplicial complex, then the vanishing of $H_1(X; \mathbb{F}_2)$ is just one way of defining “one-dimensional connectivity”. A stronger notion would be to ask for the homology group $H_1(X; \mathbb{Z})$ to vanish. Similar notions of connectivity—for arbitrary dimension and coefficients in an arbitrary finite group—have been

considered in [13]. The strongest notion of one-dimensional connectivity is to consider *simple connectivity*, i.e. the vanishing of the fundamental group. For the \mathcal{H}_p^{LM} model, this was studied in [1]. Another possibility is to consider Betti numbers [8].

For hypergraphs rather than complexes, vertex-connectivity is by far the most studied definition, and the connectivity threshold of $\frac{\log n}{\binom{n}{k}}$ in a $(k+1)$ -uniform hypergraph and a corresponding hitting time result can easily be proved analogously to the graph case. (For $k=2$ this is proved in this paper, albeit not with the sharpest possible threshold, in Lemmas 9 and 10).

More generally, for $1 \leq j \leq k$ and a $(k+1)$ -uniform hypergraph we may define a higher-order notion of connectivity on the j -sets (the case $j=1$ corresponds to vertex connectivity). Then the threshold for connectivity is $\frac{j \log n}{\binom{n}{k+1-j}}$, as proved in [3].

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